Using Branch-and-Price to Solve Multicommodity *k*-Splittable Flow Problems

Jérôme Truffot, Christophe Duhamel, Philippe Mahey

{jerome.truffot,christophe.duhamel,philippe.mahey}@isima.fr

LIMOS, UMR 6158 - CNRS
Université Blaise Pascal
Clermont-Ferrand - France

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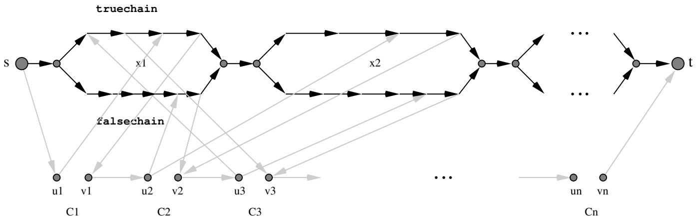
Introduction

- New technologies: optical network, MPLS protocol, . . .
- MPLS-TE: Traffic Engineering, Quality of Service
- Bounded number of active paths
- Maximum Flow Problem, Minimal Cost Flow Problem, . . .

Outline of the talk

Complexity

- SAT Reduction of 2-Splittable Flow Problem (Baier, Köhler and Skutella, 2002)



- Bounded number of paths → NP-complete problems

State of the Art: K-SFP

- Edge Disjoint Path Problem (Survivability): K. Menger (1927)
- Unsplittable Flow Problem: J.M. Kleinberg (1996), A. Atamtürk (2000)
- K-Splittable Flow Problem: G. Baier, E. Köhler et M. Skutella (2002)

Definitions

- Directed graph $G = (V, E)$
  - arc capacity u_e
  - arc cost c_e
- commodity $k \in K$: (s_k, t_k, d_k)
- maximum number of paths: H_k

Model for the Unsplittable Flow Problem

- Path support variables
- Flow = demand
- Flow constraints sufficient

$\Rightarrow$ Integer linear program

Basic Arc-Node Model

- Flow variables $x_e^k \geqslant 0$
- Flow support variables $y_e^k \in \{0, 1\}$

$\Rightarrow$ How can we add a “H_k-splittable” constraint ?

$\Rightarrow$ Consider separately H_k paths

Arc-Node Model: variables

- Backward arc $\overline{e}_k = (t_k, s_k)$, with $c_{\overline{e}_k} = 0$ and $u_{\overline{e}_k} = \infty$
- $E' = E \cup \{\overline{e}_k, k \in K\}$
- $\forall k \in K, \quad \forall h \in \{1, \dots, H_k\}$:
  - flow variables: $x_e^{hk} \geq 0 \quad \forall e \in E'$
  - flow support variables: $y_e^{hk} \in \{0, 1\} \quad \forall e \in E'$
- Objective function: max $\sum_{k \in K} \sum_{h=1}^{H_k} x_{e_k}^{hk}$

Arc-Node Model: flow constraints

- Flow conservation constraint:

$$Ax^{hk} = 0 \qquad \forall k \in K, \quad \forall h \in \{1, .., H_k\}$$

- Capacity constraint:

$$\sum_{k \in K} \sum_{h=1}^{H_k} x_e^{hk} \leq u_e \qquad \forall e \in E$$

- Coupling constraint:

$$x_e^{hk} \leq u_e y_e^{hk} \qquad \forall e \in E \quad \forall k \in K, \quad \forall h \in \{1, \ldots, H_k\}$$

Arc-Node Model: flow support constraints

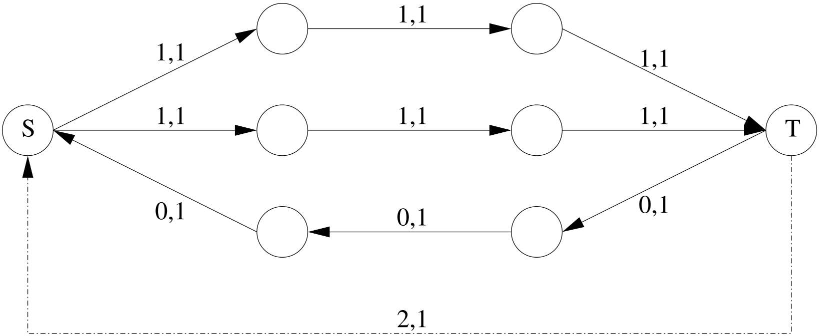
- Flow conservation constraint:

$$Ay^{hk} = 0 \qquad \forall k \in K, \quad \forall h \in \{1, \ldots, H_k\}$$

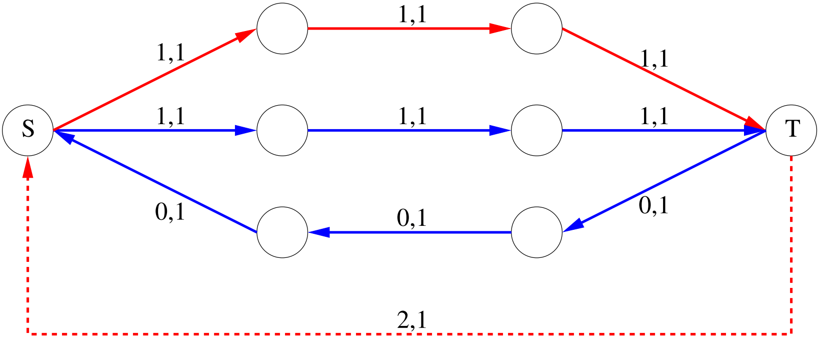
- Integrality constraint:

$$y_e^{hk} \in \{0, 1\} \qquad \forall e \in E, \quad \forall k \in K, \quad \forall h \in \{1, \ldots, H_k\}$$

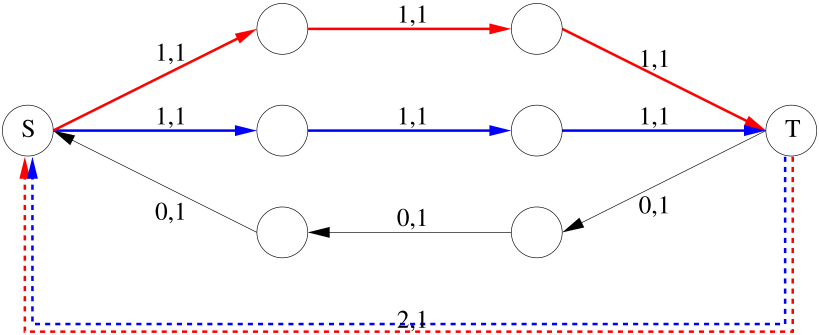
Arc-Node Model: drawback



Arc-Node Model: drawback



Arc-Node Model: drawback

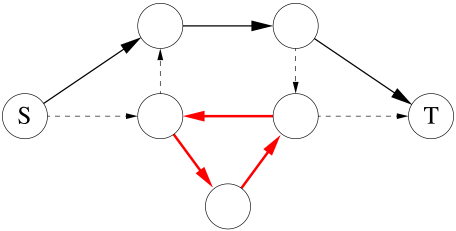


Arc-Node Model: enforcing constraints

- Degree constraint:

$$\sum_{e \in \omega^+(v)} y_e^{hk} \leq 1 \qquad \forall v \in V, \quad \forall k \in K, \quad \forall h \in \{1, \ldots, H_k\}$$

- do not forbid subcycles



Arc-Node Model:

$$\max \sum_{k \in K} \sum_{h=1}^{H_k} x_{e_k}^{hk}$$

s.c.

$$
\begin{array}{llr}
Ax^{hk} = 0 & \forall k \in K, \quad \forall h \in \{1, \dots, H_k\} & (1.a) \\
Ay^{hk} = 0 & \forall k \in K, \quad \forall h \in \{1, \dots, H_k\} & (1.b) \\
\sum_{k \in K} \sum_{h=1}^{H_k} x_e^{hk} \leq u_e & \forall e \in E & (1.c) \\
x_e^{hk} - u_e y_e^{hk} \leq 0 & \forall e \in E, \quad \forall k \in K, \quad \forall h \in \{1, \dots, H_k\} & (1.d) \\
\sum_{e \in \omega^+(v)} y_e^{hk} \leq 1 & \forall v \in V, \quad \forall k \in K, \quad \forall h \in \{1, \dots, H_k\} & (1.e) \\
x_e^{hk} \geq 0 & \forall e \in E, \quad \forall k \in K, \quad \forall h \in \{1, \dots, H_k\} & (1.f) \\
y_e^{hk} \in \{0, 1\} & \forall e \in E, \quad \forall k \in K, \quad \forall h \in \{1, \dots, H_k\} & (1.g)
\end{array}
$$

Arc-Path Model: variable

- P_k = set of loopless paths from s_k to t_k
- $\forall k \in K, \quad \forall h \in \{1, \ldots, H_k\}$:
  - flow variables: $x_p^{hk} \geq 0 \quad \forall p \in P_k$
  - flow support variables: $y_p^{hk} \in \{0, 1\} \quad \forall p \in P_k$
- Objective function: max $\sum_{k \in K} \sum_{h=1}^{H_k} \sum_{p \in P_k} x_p^{hk}$

Arc-Path Model: constraints

- Capacity constraint:

$$\sum_{k \in K} \sum_{h=1}^{H_k} \sum_{p \in P^k} \delta_e^p x_p^{hk} \leq u_e \qquad \forall e \in E$$

- Coupling constraint:

$$x_p^{hk} \leq u_p y_p^{hk} \qquad \forall k \in K, \quad \forall h \in \{1, \ldots, H_k\}, \quad \forall p \in P_k$$

path capacity $u_p = \min_{e \in p} u_e$

- Path assignment constraint:

$$\sum_{p \in P_k} y_p^{hk} \leq 1 \qquad \forall k \in K, \quad \forall h \in \{1, \ldots, H_k\}$$

Arc-Path Model:

$$\max \sum_{k \in K} \sum_{h=1}^{H_k} \sum_{p \in P_k} x_p^{hk}$$

$s.c.$

$$\sum_{h,k,p} \delta_e^p x_p^{hk} \leq u_e \qquad \forall e \in E \qquad (2.a)$$

$$x_p^{hk} - u_p y_p^{hk} \leq 0 \qquad \forall k \in K \quad \forall h \in \{1, \ldots, H_k\} \quad \forall p \in P_k \qquad (2.b)$$

$$\sum_{p \in P_k} y_p^{hk} \leq 1 \qquad \forall k \in K \quad \forall h \in \{1, \ldots, H_k\} \qquad (2.c)$$

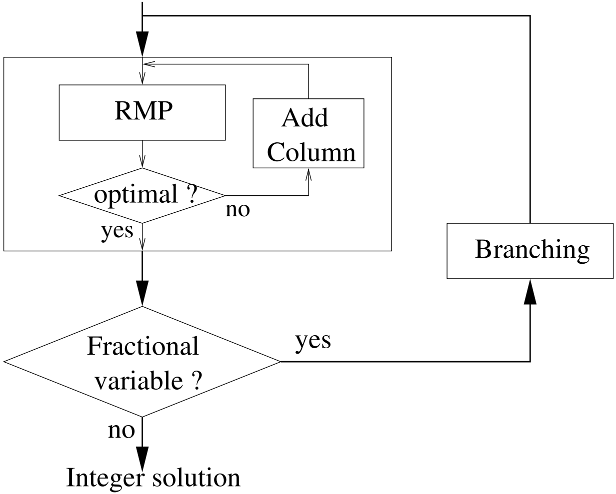
$$x_p^{hk} \geq 0 \qquad \forall k \in K \quad \forall h \in \{1, \ldots, H_k\} \quad \forall p \in P_k \qquad (2.d)$$

$$y_p^{hk} \in \{0, 1\} \qquad \forall k \in K \quad \forall h \in \{1, \ldots, H_k\} \quad \forall p \in P_k \qquad (2.e)$$

State of the art: Branch & Price

- Branch & Bound: A. Land et A. Doig (1960)
- Branch & Price: C. Barnhart, E.L. Johnson, G.L. Nemhauser, M.W.P. Savelsbergh, P.H. Vance (1998)
- Branch & Price for the UFP:
  - C. Barnhart, C.A. Hane, P.H. Vance (2000)
  - F. Alvelos et J.M. Valério de Carvalho (2003)

Branch & Price: C. Barnhart & al. (1998)



Ryan & Foster branching rule (1981)

- Branching on variables of the arc-node model:

$$(y_e^{hk} = 0) \quad \text{vs.} \quad (y_e^{hk} = 1)$$

Ryan & Foster branching rule (1981)

- Branching on variables of the arc-node model:

$$(y_e^{hk} = 0) \quad \text{vs.} \quad (y_e^{hk} = 1)$$

- Ryan & Foster branching rule:
  - Constraint $\sum_{j \in J} y_j = 1$
  - Partition $J = J_1 \cup J_2$
  - Branching rule

$$(\sum_{j \in J_1} y_j = 0) \quad \text{vs.} \quad (\sum_{j \in J_1} y_j = 1)$$

Ryan & Foster branching rule (1981)

- Branching on variables of the arc-node model:

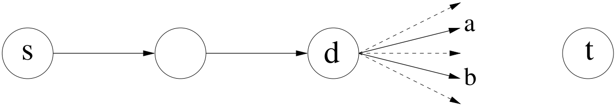
$$(y_e^{hk} = 0) \quad \text{vs.} \quad (y_e^{hk} = 1)$$

- Ryan & Foster branching rule:
  - Constraint $\sum_{j \in J} y_j = 1$
  - Partition $J = J_1 \cup J_2$
  - Branching rule

$$(\sum_{j \in J_1} y_j = 0) \text{ vs. } (\sum_{j \in J_2} y_j = 0)$$

Barnhart, Hane & Vance branching rule

- First divergence node:



- Constraint $\sum_{e \in \omega^+(d)} y_e^{hk} \leq 1$
- Partition $\omega^+(d) = \omega^+(d,a) \cup \omega^+(d,b)$
- Branching rule $\left(\sum_{e \in \omega^+(d,a)} y_e^{hk} = 0\right)$ vs. $\left(\sum_{e \in \omega^+(d,b)} y_e^{hk} = 0\right)$

Restricted Master Program

$$\max \sum_{k \in K} \sum_{h=1}^{H_k} \sum_{p \in P_k} x_p^{hk}$$

$$s.c.$$

$$\sum_{h,k,p} \delta_e^p x_p^{hk} \leq u_e \quad \forall e \quad (3.a)$$

$$x_p^{hk} - u_p y_p^{hk} \leq 0 \quad \forall k \ \forall h \ \forall p \quad (3.b) \qquad \Rightarrow y_p^{hk} = \frac{x_p^{hk}}{u_p}$$

$$\sum_{p \in P_k} y_p^{hk} \leq 1 \quad \forall k \ \forall h \quad (3.c)$$

$$x_p^{hk} \geq 0 \quad \forall k \ \forall h \ \forall p \quad (3.d)$$

$$y_p^{hk} \in [0,1] \quad \forall k \ \forall h \ \forall p \quad (3.e)$$

Reduced costs computation

$$\max \sum_{k \in K} \sum_{h=1}^{H_k} \sum_{p \in P_k} x_p^{hk}$$

s.c.

$$\sum_{h,k,p} \delta_e^p x_p^{hk} \leq u_e \qquad \forall e \qquad \rightarrow \pi_e \geq 0$$

$$\sum_p \frac{x_p^{hk}}{u_p} \leq 1 \qquad \forall k\ \forall h \qquad \rightarrow \lambda^{hk} \geq 0$$

$$x_p^{hk} \geq 0 \qquad \forall k\ \forall h\ \forall p$$

$\Rightarrow$ Reduced costs:

$$\overline{c_p^{hk}} = 1 - \sum_{e \in E} \delta_e^p \pi_e - \frac{\lambda^{hk}}{u_p}$$

Minimal Cost k-Splittable Flow Problem

- Demand Constraint and associated dual variable:

$$\sum_{h=1}^{H_k} \sum_{p \in P_k} x_p^{hk} = d_k \qquad \forall k \in K \qquad \rightarrow \nu^k \text{free}$$

- Objective function: $\min \sum_{k \in K} \sum_{h=1}^{H_k} \sum_{p \in P_k} c_p x_p^{hk}$

  with path cost $c_p = \sum_{e \in p} c_e$

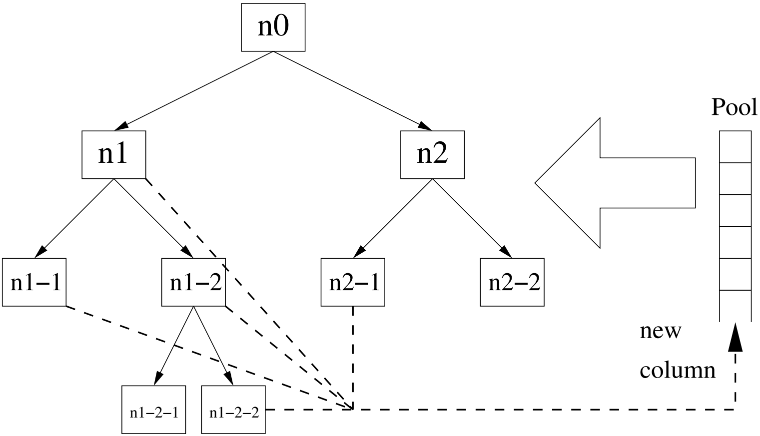
- Reduced costs:

$$\overline{c_p^{hk}} = c_p - \sum_{e \in E} \delta_e^p \pi_e - \frac{\lambda^{hk}}{u_p} - \nu^k = \sum_{e \in E} \delta_e^p (c_e - \pi_e) - \frac{\lambda^{hk}}{u_p} - \nu^k$$

Subproblem: shortest path / maximum capacity path

- Step 1: shortest path with maximum capacity c (modified Dijkstra)
- Step 2: for increasing capacity $c^t > c$
  - shortest path with at least capacity c^t
- Step 3: select the best one

Pool



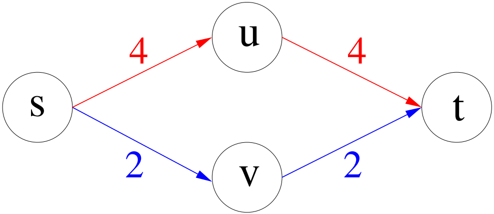
Variable Ordering

- "Break" model symmetry
- Bring an order relation over the set of paths
- Constraint :

$$\sum_{p \in P_k} x_p^{hk} - \sum_{p \in P_k} x_p^{(h-1)k} \leq 0 \qquad \forall k \in K \quad \forall h \in \{2, \ldots, H_k\}$$

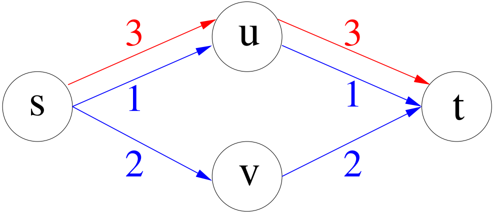
Variable Ordering: drawback

- Without variable ordering



Variable Ordering: drawback

- With variable ordering



- Variable ordering may give split solutions

Computationnal Results

- Instances :
  - `ac` generator : acyclic network, single commodity (G. Waissi)
  - `tg` generator : transid grid, single commodity (G. Waissi and J. Setubal)
- Processor pentium 2.6GHz, linux 2.6, 2Go RAM
- Compiler gcc 3.3, ILOG CPLEX 8.0

CPU Time Comparison

Graph	H	z^*	CPU Time (s)			
			C	BP	BP-P	BP-VP
ac100	1	9827.00	7.85	0.06	0.08	0.08
100-4950	2	19403.00	570.09	0.08	0.09	0.09
	3	28809.00	-	0.10	0.10	0.09
	4	38189.00	-	0.10	0.14	0.13
	5	47558.00	-	0.12	0.17	0.19
	6	56791.00	-	735.07	111.52	1.79
	7	65838.00	-	-	2426.31	8.81
	8	74762.00	-	-	-	36.67
	9	83501.00	-	-	-	151.00
	10	92170.00	-	-	-	1270.90
	∞	480094.00				

CPU Time Comparison

Graph	H	z^*	CPU Time (s)			
			C	BP	BP-P	BP-VP
ac125	1	9650.00	188.04	-	0.13	0.13
125-7750	2	19298.00	1318.50	-	0.15	0.15
	3	28864.00	-	-	0.17	0.17
	4	38374.00	-	-	0.21	0.22
	5	47725.00	-	-	0.28	0.33
	6	57007.00	-	-	162.15	5.98
	7	66241.00	-	-	-	15.86
	8	75320.00	-	-	-	55.40
	9	84396.00	-	-	-	161.56
	10	93286.00	-	-	-	795.04
	11	102175.00	-	-	-	1758.78
	∞	584842.00				

CPU Time Comparison

Graph	H	z^*	CPU Time (s)			
			C	BP	BP-P	BP-VP
tg25	1	236.00	2.15	0.00	0.00	0.00
27-100	2	459.00	-	0.00	0.01	0.00
	3	673.00	-	0.03	0.01	0.01
	4	875.00	-	0.06	0.01	0.10
	5	961.00	-	1397.79	191.90	30.08
	6	972.00	-	-	-	1940.99
	7	980.00	-	-	-	3.14
	∞	980.00				

CPU Time Comparison

Graph	H	z^*	CPU Time (s)			
			C	BP	BP-P	BP-VP
tg100	1	550.00	1706.20	0.01	0.01	0.01
102-400	2	1012.00	-	0.20	0.03	0.04
	3	1396.00	-	538.71	578.44	68.18
	∞	2759.00				

Concluding Remarks

- Generalization of the Unsplittable Flow Problem
- Branch & Price scheme, variable ordering, pool

- Improve the variable ordering technique

- Other problems:
  - Maximal Concurrent Flow
  - Minimal Set of Paths for a Flow
  - Minimal Set of LSP in MPLS routing with QoS constraint